Comparison Game On Trace Ideals

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Definition

Let X be a countable set and $\mathcal{I} \subseteq \mathcal{P}(X)$, $\mathcal{I}$ is called an **ideal** if:

(1). $[X]^{<\omega} \subseteq \mathcal{I}$.

(2). $B \in \mathcal{I}, A \subseteq B \Longrightarrow A \in \mathcal{I}$.

(3). $A, B \in \mathcal{I} \Longrightarrow A \cup B \in \mathcal{I}$.

An ideal $\mathcal{I}$ is called P-**ideal** if

(4). $\forall \{B_n : n \in \omega\} \subseteq \mathcal{I}$, $\exists B \in \mathcal{I}(\forall n \in \omega B_n \subseteq^* B)$.

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Definition (M. Hrušák and D. M. Alcántara)

Let $\mathcal{I}$ and $\mathcal{J}$ be ideals on ω. The **Comparison Game** for $\mathcal{I}$ and $\mathcal{J}$ denoted by $G(\mathcal{I}, \mathcal{J})$ is played as follow:

I	$I_0 \in \mathcal{I}$		$\cdots$	$I_n \in \mathcal{I}$		$\cdots$
II		$J_0 \in \mathcal{J}$	$\cdots$		$J_n \in \mathcal{J}$	$\cdots$

Player II wins if $\bigcup_{n\in\omega} I_n \in \mathcal{I} \Longleftrightarrow \bigcup_{n\in\omega} J_n \in \mathcal{J}$.

We write $\mathcal{I} \sqsubseteq \mathcal{J}$ if Player II has a winning strategy in $G(\mathcal{I}, \mathcal{J})$. And $\mathcal{I} \simeq \mathcal{J}$ if $\mathcal{I} \sqsubseteq \mathcal{J} \wedge \mathcal{J} \sqsubseteq \mathcal{I}$.

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Motivation

Let $\mathcal{I}$ and $\mathcal{J}$ be two ideals on ω.
A function $f : \mathcal{I} \longrightarrow \mathcal{J}$ is called **Tukey** if

$$\forall A \in \mathcal{J} \exists B \in \mathcal{I} \forall C \in \mathcal{I}(f(C) \subseteq A \Rightarrow C \subseteq B).$$

We write

$$\mathcal{I} \leq_{MT} \mathcal{J}$$

if there is a monotone Tukey function from $\mathcal{I}$ to $\mathcal{J}$.

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Definition (W. Wadge)

Let $X, Y \subseteq \omega^\omega$. The **Wadge Game** for X and Y denoted by $W(X,Y)$ is played as follow:

I	$x_0 \in \omega$		$\cdots$	$x_n \in \omega$		$\cdots$
II		$y_0 \in \omega$	$\cdots$		$y_n \in \omega$	$\cdots$

Denote $x = x_0 x_1 ... x_n ...$ and $y = y_0 y_1 ... y_n ...$.
Player II wins if $x \in X \iff y \in Y$.

We write $X \leq_W Y$ if Player II has a winning strategy in $W(X,Y)$. And $X \equiv_W Y$ if $X \leq_W Y \wedge Y \leq_W X$.

Theorem (M.Hrušák, D. M. Alcántara)

$\mathcal{I} \sqsubseteq \mathcal{J} \iff \tilde{\mathcal{I}} \leq_W \tilde{\mathcal{J}}$, where $\tilde{\mathcal{I}} = \{x \in \omega^\omega : rang(x) \in \mathcal{I}\}$.

Corollary

- *The game $G(\mathcal{I}, \mathcal{J})$ is determined for every pair $\mathcal{I}, \mathcal{J}$ of Borel ideals.*
- *The order $\sqsubseteq$ is well-founded.*
- *The comparison game is almost linear (all antichains have size at most 2).*
- *There are uncountable many $\simeq$-classes.*

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Theorem (M.Hrušák, D. M. Alcántara)

(1). *For any Borel ideal* $\mathcal{I}$. $\mathcal{I}$ *is* F_σ *if and only if* $\mathcal{I} \simeq Fin$.

(2). *There are at least two classes of* $F_{\sigma\delta}(\mathbf{\Pi}_3^0)$ *non-*$F_\sigma(\mathbf{\Sigma}_2^0)$*-ideals.*

(3). *Let* $\mathcal{I}$ *be an analytic* P*-ideal. Then* $\mathcal{I} \simeq Fin$ *or* $\mathcal{I} \simeq \emptyset \times Fin$.

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Question (M.Hrušák, D. M. Alcántara)

(1). *Is the order $\sqsubseteq$ linear?*

(2). *Are there exactly two class of $F_{\sigma\delta}$ non F_σ-ideals?*

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Definition

Let X be a Borel subset of 2^ω. The **trace ideal** of X, denoted by $T(X)$, is the ideal on ${}^{<\omega}2$ generated by $\{\{x|n : n \in \omega\} : x \in X\}$.

Proposition (van. Engelen 1994)

Let $\Gamma \supseteq \Delta(D_\omega(\Sigma^0_2))$ *be a Wadge degree such that* $\forall n \in \omega \forall X \in \Gamma \Rightarrow X^n \in \Gamma$. *If* X *is* Γ, *then* $T(X)$ *is* Γ.

A subset $A \subseteq 2^\omega$ is $D_\omega(\mathbf{\Sigma}^0_2)$ if there is a increasing $\mathbf{\Sigma}^0_2$ sequence $\{B_n : n \in \omega\}$ such that $A = \bigcup_{k\in\omega} B_{2k+1} \setminus B_{2k}$.

Lemma (van. Engelen)

If $\mathcal{I}$ *is an infinite Borel ideal on* ω, *then* $\mathcal{I} \times \mathcal{I} \equiv_W \mathcal{I}$.

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Observation: Two $F_{\sigma\delta}$ non-F_σ-ideals: $\emptyset \times Fin$, $T(Fin^+)$.
We have the following result:

Theorem

(1). $\emptyset \times Fin \not\sqsubseteq T((\emptyset \times Fin)^+)$. *Where*
$\emptyset \times Fin = \{A \subseteq \omega \times \omega : \forall n \in \omega |\{m : (n, m) \in A\}| < \omega\}$.

(2). $T((\emptyset \times Fin)^+) \not\sqsubseteq \emptyset \times Fin$.

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Lemma (J.Steel)

Let Γ be a Wedge class above $D_2(\Sigma^0_2)$. Then $\forall A, B((A \in \Gamma \wedge B \in \Gamma \setminus \check{\Gamma}) \Rightarrow A \leq_1 B)$. Where $A \leq_1 B$ means there is injection continuous $f : 2^\omega \longrightarrow 2^\omega$ such that $A = f^{-1}(B)$.

The key idea: If the result of Player I plays has finite many anti-chains, use $X \leq_W Y$ control.
If it has infinite many anti-chains, use 1-1 to preserve the result of Player II also has infinite many anti-chains.

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Theorem

Let $A \subseteq 2^\omega$ be a Borel subset such that its Wadge class is above $D_\omega(\mathbf{\Sigma}^0_2)$ and B be any Borel set. If $B^c \not\leq_W A$, then $T(T(B)^+) \not\sqsubseteq T(A)$.

Corollary

If $\mathcal{I}, \mathcal{J}$ be two Borel ideals above $D_\omega(\Sigma^0_2)$, then $\mathcal{I} \equiv_W \mathcal{J} \Leftrightarrow T(\mathcal{I}) \simeq T(\mathcal{J})$.

Are there exactly two class of $F_{\sigma\delta}$ non F_σ-ideals?
How many classes of $F_{\sigma\delta\sigma}$-ideals are there?
No, there are ω_1 many classes.

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$T(\mathcal{I})$ vs. $\mathcal{I}$

Theorem

Let $\mathcal{I}$ be a Borel ideal. Then $T(\mathcal{I}) \sqsubseteq \mathcal{I}$.

But we don't clear whether $T(\mathcal{I}) \simeq \mathcal{I}$,
Form above theorem we have that:

Corollary

For any Borel ideal $\mathcal{I}$ with Wadge class above $D_\omega(\Sigma_2^0)$, we have $T(T(\mathcal{I})) \simeq T(\mathcal{I})$.

$T(\mathcal{I})$ vs. $\mathcal{I}$

Theorem

Let $\mathcal{I}$ be a Borel ideal. Then $T(\mathcal{I}) \sqsubseteq \mathcal{I}$.

But we don't clear whether $T(\mathcal{I}) \simeq \mathcal{I}$,
Form above theorem we have that:

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Definition

For every $0 < \mu < \omega_1$, we let $Fr_{2\mu} = \{S \subseteq \omega^\mu : |S|_L < \omega^\mu\}$, $Fr_{2\mu+1} = \{S \subseteq \omega^{\mu+1} : \forall m \in \omega((S)_m \in Fr_{2\mu})\}$.

Theorem

For every $0 < \mu < \omega_1$, $T(Fr_{2\mu}) \simeq Fr_{2\mu}$, $T(Fr_{2\mu+1}) \simeq Fr_{2\mu+1}$.

Corollary

Let $\mathcal{I}$ be a Borel ideal. If $Fr_{2\mu} \leq_W \mathcal{I}$, then $Fr_{2\mu} \sqsubseteq \mathcal{I}$ and if $Fr_{2\mu+1} \leq_W \mathcal{I}$, then $Fr_{2\mu+1} \sqsubseteq \mathcal{I}$.

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A dichotomy for analytic P-ideal

Theorem (M.Hrušák, D. M. Alcántara)

Let $\mathcal{I}$ be an analytic P-ideal. Then $\mathcal{I} \simeq Fin$ or $\mathcal{I} \simeq \emptyset \times Fin$.

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Let $\mathcal{I}$ be an analytic P-ideal. If $\mathcal{I}$ is non-F_σ, then $\mathcal{I}$ is $F_{\sigma\delta}$-complete.

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Theorem

$\emptyset \times Fin \not\sqsubseteq T((\emptyset \times Fin)^+)$

The idea of proof.

We construct a winning strategy for Player I in $G(\emptyset \times Fin, T((\emptyset \times Fin)^+))$.

The possible case 1: The maximal anti-chains of result of Player II is finite.

For every $1 \leq n < \omega$ we define a game G_n as follows. In step k, Player I pick a $I_k \in \emptyset \times Fin$ and Player II picks a $J_k \in T((\emptyset \times Fin)^+)$ such that $\forall i < k(J_i \subseteq J_k)$ and the maximal cardinal of an antichain of J_k is n. Player II wins if $\bigcup_{n\in\omega} I_n \in \emptyset \times Fin$ iff $\bigcup_{n\in\omega} J_n \in T((\emptyset \times Fin)^+)$.

Claim: Player I have a winning strategy σ_n in G_n. (Proved by induction on n)

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Copy $\emptyset \times Fin$ ω many times by the structure of $\emptyset \times Fin$.

The strategy of Player I as follows:
Let $\{X_n : n \in \omega\} \subseteq [\omega]^\omega$ be a partition of ω. In step 0, Play I plays $\emptyset$, and in step $k > 0$, Let $M(k)$ be the maximal cardinal of an antichain in $\bigcup_{i<k} J_i$.
(1) If $M(k) = M(k-1)$, then Player I plays the game $G_{M(k-1)}$ in $X_{M(k-1)} \times \omega$ think Player II plays $\bigcup_{i<k} J_i$ in a new step and plays follow the winning strategy $\sigma_{M(k-1)}$.
(2) If $M(k) > M(k-1)$, then Player I has to abandon what he has played and begin a new game of $G_{M(k)}$ inside $X_{M(k)} \times \omega$ follow the winning strategy $\sigma_{M(k)}$, and think Player II in step 0 plays $\bigcup_{i<k} J_i$. □

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Děkuji!